

# LIFTING OF NILPOTENT CONTRACTIONS

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## INTRODUCTION

For the standard epimorphism from a  $C^*$ -algebra  $A$  to its quotient  $A/I$  by a closed ideal  $I$ , one may ask whether an element  $b$  in  $A/I$  with some specific properties is the image of some element  $a$  in  $A$  with the same properties. This is known as a lifting problem that can be considered as a non-commutative analogue of extension problems for functions.

In the present work we deal with the following lifting problem. Let  $A$  be any  $C^*$ -algebra,  $A/I$  — any its quotient,  $b \in A/I$  — a nilpotent contraction, that is  $b^n = 0$ , for some fixed number  $n$ , and  $\|b\| \leq 1$ . Find an element  $a \in A$  that is a counter-image of  $b$  and is also a nilpotent contraction:  $a^n = 0$  and  $\|a\| \leq 1$ .

The problem can be reformulated in terms of representations of relations. For the correct definition of a relation (or a system of relations) see [1]. Vaguely speaking a relation is an equality of the form  $f(x_1, \dots, x_n, x_1^*, \dots, x_n^*) = 0$  where  $f$  is a non-commutative polynomial (an element of the corresponding free algebra) and also inequalities of the form  $\|x_i\| \leq c$  or  $\|x_i\| < c$  are admitted. By a representation of a relation in a  $C^*$ -algebra  $A$  one means an  $n$ -tuple  $(a_1, \dots, a_n)$  of elements of  $A$  that satisfy with their adjoints the given equality and inequalities (if any).

A relation is liftable if for each its representation  $(b_1, \dots, b_n)$  in a quotient algebra  $A/I$ , there is a representation  $(a_1, \dots, a_n)$  in  $A$  such that  $b_i = a_i + I$ .

So the question we consider is

**Problem 1.** *Are the relations*

$$x^n = 0 \text{ and } \|x\| \leq 1 \quad (1)$$

*liftable?*

It was proved in [3] that each nilpotent element can be lifted to a nilpotent element. Later in [1] it was shown that among nilpotent preimages of a nilpotent element  $b$  there are elements of norm  $\|b\| + \epsilon$ , for any  $\epsilon > 0$ . This implies that the relations

$$x^n = 0 \text{ and } \|x\| < 1 \quad (2)$$

are liftable.

Problem 1 was stated in [1] in connection with the study of projective  $C^*$ -algebras. A  $C^*$ -algebra  $B$  is projective if for any homomorphism  $f$  from  $B$  to a quotient  $A/I$ , there is a homomorphism  $g : B \rightarrow A$  with  $f(b) = g(b) + I$ ,  $b \in B$ . Projective  $C^*$ -algebras were introduced in [4]. There are very few  $C^*$ -algebras that are known to be projective ([1], [6]).

A  $C^*$ -algebra  $\mathcal{A}$  is called a universal  $C^*$ -algebra of a relation if there is a representation  $\pi_0$  of this relation in  $\mathcal{A}$  such that any representation of the relation factorizes uniquely through  $\pi_0$ . Note that not every system of relations has the universal  $C^*$ -algebra, for

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*Date:* February 2, 2008.

*2000 Mathematics Subject Classification.* 46 L05; 46L35.

*Key words and phrases.* Projective  $C^*$ -algebra, lifting problem, nilpotent element, M-ideal.

example the relations (2) don't have it (the reason is that an infinite direct sum of strict contractions needn't be a strict contraction).

It is not difficult to see that the universal  $C^*$ -algebra of a relation is projective if and only if the relation is liftable. So the question can be reformulated as follows: is the universal  $C^*$ -algebra of the relations (1) projective?

The aim of the present work is to give the positive answer to Problem 1 and as a consequence to extend the list of known projective  $C^*$ -algebras.

## 1. M-IDEALS

We recall some geometrical definitions (see, e.g., [2]).

A closed subspace  $Y$  of a Banach space  $X$  is called an L-summand (M-summand), if  $X = Y \oplus Z$  for some subspace  $Z \subset X$ , and  $\|y + z\| = \|y\| + \|z\|$  (respectively  $\|y + z\| = \max\{\|y\|, \|z\|\}$ ) for all  $y \in Y$ ,  $z \in Z$ . Furthermore  $Y$  is an M-ideal in  $X$  if its annihilator  $Y^\perp$  is an L-summand of  $X^*$ .

Let  $T$  be an operator on a Banach space  $X$ . Let us say that an M-ideal  $Y \subset X$  *dually reduces*  $T$  if both summands of the L-decomposition  $X^* = Y^\perp + Z$  are invariant for  $T^*$ . The definition is correct because a subspace can have only one L-complement. Indeed if  $X = Y + Z$  and  $X = Y + W$  are L-decompositions then for  $z \in Z$  one has  $z = y + w$  and  $w = z + (-y)$  whence  $\|z\| = \|z\| + 2\|y\|$ ,  $y = 0$ ,  $z \in W$ .

**Theorem 2.** *If an M-ideal  $Y$  of  $X$  dually reduces an operator  $T$  then  $\overline{TY}$  is an M-ideal of  $\overline{TX}$ .*

*Proof.* For any subspace  $E \subset X$ , one has the canonical isometrical isomorphism  $i : E^* \rightarrow X^*/E^\perp$ . Namely, for any  $h \in E^*$ ,  $i(h) = \tilde{h} + E^\perp$ , where  $\tilde{h}$  is any extension of  $h$  onto  $X$ . Hence

$$i(\overline{TX}^*) = X^*/(TX)^\perp = X^*/\ker(T^*).$$

Let  $q$  denote the quotient map from  $X^*$  to  $X^*/\ker(T^*)$ .

We have the L-sum decomposition  $X^* = Y^\perp \oplus Z$ ; by the assumption, both summands are  $T^*$ -invariant.

Let us prove that the decomposition  $q(X^*) = q(Y^\perp) + q(Z)$  is an L-sum, that is for any  $\xi = \eta + \zeta$ , where  $\eta \in q(Y^\perp)$ ,  $\zeta \in q(Z)$ , the equality  $\|\xi\| = \|\eta\| + \|\zeta\|$  holds.

For any  $\varepsilon > 0$ , there is  $f \in X^*$  such that  $\xi = q(f)$  and  $\|\xi\| \geq \|f\| - \varepsilon$ . There are  $f_1 \in Y^\perp$ ,  $f_2 \in Z$  such that  $f = f_1 + f_2$ . Prove that  $q(f_1) = \eta$ ,  $q(f_2) = \zeta$ . Consider  $g_1 \in Y^\perp$ ,  $g_2 \in Z$  such that  $\eta = q(g_1)$ ,  $\zeta = q(g_2)$ . Then

$$q(g_1 + g_2) = \eta + \zeta = \xi = q(f) = q(f_1 + f_2),$$

whence  $g_1 - f_1 + g_2 - f_2 \in \ker(T^*)$ . Hence  $T^*(g_1 - f_1) = T^*(-g_2 + f_2)$ . Since  $Y^\perp$  and  $Z$  are invariant subspaces for  $T^*$ , we get  $T^*(g_1 - f_1) \in Y^\perp$ ,  $T^*(-g_2 + f_2) \in Z$ , whence  $T^*(g_1 - f_1) = T^*(-g_2 + f_2) = 0$ . Hence  $\eta = q(g_1) = q(f_1)$ ,  $\zeta = q(g_2) = q(f_2)$  and we have

$$\|\xi\| \geq \|f\| - \varepsilon = \|f_1\| + \|f_2\| - \varepsilon \geq \|q(f_1)\| + \|q(f_2)\| - \varepsilon = \|\eta\| + \|\zeta\| - \varepsilon.$$

Since it is true for every  $\varepsilon > 0$  and since  $\|\xi\| \leq \|\eta\| + \|\zeta\|$ , the equality  $\|\xi\| = \|\eta\| + \|\zeta\|$  holds. In particular it easily follows that  $q(Y^\perp) \cap q(Z) = 0$  and both subspaces  $q(Y^\perp)$  and  $q(Z)$  are closed.

Thus  $q(Y^\perp)$  is an L-summand in  $q(X^*)$ . It remains to prove that  $q(Y^\perp) = i((\overline{TY})^\perp)$ , where by  $\overline{TY}^\perp$  we denote the annihilator of  $TY$  in  $\overline{TX}^*$ . Let  $f \in Y^\perp$  and  $f_1$  be its restriction to  $\overline{TX}$ . Then, by definition,  $q(f) = i(f_1)$ . For each  $y \in Y$ , one has  $f_1(Ty) = f(Ty) = (T^*f)(y) = 0$  because  $T^*$  preserves  $Y^\perp$ . Hence  $f_1$  belongs to  $\overline{TY}^\perp$ . We proved that  $q(Y^\perp) \subset i(\overline{TY}^\perp)$ .

Conversely, let  $h \in (\overline{TY})^\perp$ . There are  $f_1 \in Y^\perp$ ,  $f_2 \in Z$  such that  $i(h) = q(f_1) + q(f_2)$ . Let  $\tilde{h}$  be any extension of  $h$  onto  $X$ . Then  $\tilde{h} - f_1 - f_2 \in \ker(T^*)$  and  $T^*\tilde{h} - T^*f_1 - T^*f_2 = 0$ . Since, for any  $y \in Y$ ,  $(T^*\tilde{h})(y) = \tilde{h}(Ty) = h(Ty) = 0$ ,  $T^*\tilde{h} \in Y^\perp$ . Since  $T^*\tilde{h} - T^*f_1 \in Y^\perp$  and  $T^*f_2 \in Z$ , we get  $T^*f_2 = 0$ , that is  $q(f_2) = 0$ . Hence  $i(h) \in q(Y^\perp)$  and  $q(Y^\perp) = i((\overline{TY})^\perp)$ .

We get that  $i((\overline{TY})^\perp)$  is an L-summand in  $i((\overline{TX})^*)$ , or, equivalently, that  $\overline{TY}$  is an M-ideal of  $\overline{TX}$ .  $\square$

For an arbitrary algebra  $A$ , an operator from  $A$  to  $A$  is called an elementary operator if it is of the form  $x \mapsto \sum_{i=1}^N a_i x b_i$ , where  $a_i, b_i \in A$ .

**Theorem 3.** *Let  $A$  be a  $C^*$ -algebra,  $I$  — its ideal,  $T : A \rightarrow A$  — an elementary operator. Then  $\overline{TI}$  is an M-ideal in  $\overline{TA}$ .*

*Proof.* Recall first of all that the bidual  $A^{**}$  of  $A$  is a  $W^*$ -algebra. Moreover the natural  $A$ -bimodule structure in  $A^*$  ( $(af)(x) = f(xa)$ ,  $(fa)(x) = f(ax)$ ) extends by the  $*$ -weak continuity to  $A^{**}$ , so  $A^*$  becomes an  $A^{**}$ -bimodule.

We may assume that the ideal  $I$  is closed. It is well known ([5]) that  $I$  is an M-ideal in  $A$ . Moreover there exists a central projection  $p \in A^{**}$  such that  $I^\perp = pA^*$  and  $A^* = pA^* + (1 - p)A^*$  is an L-sum. Since  $T$  is elementary,  $T^*$  commutes with the multiplication by  $p$ , so both summands are invariant for  $T^*$ . This means that  $I$  dually reduces  $T$ . It remains to apply Theorem 2.  $\square$

A subspace  $Y$  of a Banach space  $X$  is called proximal (e.g. [2]) if for any  $x \in X$  there is  $y \in Y$  such that  $\|x - y\| = d(x, Y)$ , or, equivalently, if for any  $z \in X/Y$  there is a lift  $x$  of  $z$  such that  $\|x\| = \|z\|$ .

**Corollary 4.** *Let  $A$  be a  $C^*$ -algebra,  $T$  an elementary operator on  $A$ . Then, for each ideal  $I$  of  $A$ , the subspace  $\overline{TI}$  is proximal in  $\overline{TA}$ .*

*Proof.* By ([2], Theorem II.1.1), all M-ideals are proximal subspaces. So it suffices to use Theorem 3.  $\square$

## 2. PROJECTIVITY

Below we use the following notation.

By  $M(I)$  we denote the multiplier  $C^*$ -algebra of a  $C^*$ -algebra  $I$ .

For elements  $x, y$  of a  $C^*$ -algebra  $A$ ,  $x \ll y$  means  $xy = yx = x$ .

For a closed ideal  $I$  of  $A$  we denote by  $a \rightarrow \dot{a}$  the standard epimorphism from  $A$  to  $A/I$ . We say that  $a$  is a lift of  $b$  if  $\dot{a} = b$ .

**Theorem 5.** *The universal  $C^*$ -algebra of the relations (1) is projective.*

*Proof.* By Theorem 10.1.9 of [1], it suffices to prove that any  $*$ -homomorphism from the universal  $C^*$ -algebra of the relations (1) to any quotient  $M(I)/I$  lifts to a  $*$ -homomorphism to  $M(I)$ , or, equivalently, any nilpotent contraction in  $M(I)/I$  lifts to a nilpotent contraction in  $M(I)$ .

Fix a nilpotent contraction  $b$  in  $M(I)/I$ :  $b^n = 0$ . If  $\|b\| < 1$  then it can be lifted to a nilpotent contraction by Theorem 12.1.6 of [1]. Thus we assume  $\|b\| = 1$ . It is proved in [3] that there are elements

$$0 \leq p_{n-1} \ll q_{n-1} \ll p_{n-2} \ll \dots \ll q_2 \ll p_1 \ll q_1 \leq q_0 = 1$$

in  $M(I)$  such that  $\sum_{j=1}^{n-1}(q_{j-1} - q_j)ap_j$  is a nilpotent (of order  $n$ ) lift of  $b$ , for any lift  $a$  of  $b$ . Let

$$E = \left\{ \sum_{j=1}^{n-1} (q_{j-1} - q_j)ap_j \mid \dot{a} = b \right\}.$$

Note that  $\overline{E}$  also consists of nilpotent lifts of  $b$ . Indeed if  $a_k \rightarrow a$ ,  $a_k^n = 0$ ,  $\dot{a}_k = b$  then  $a^n = 0$  and  $\dot{a} = b$ . We are going to find in  $\overline{E}$  an element of norm 1. Define an operator  $T : M(I) \rightarrow M(I)$  by

$$Tx = \sum_{j=1}^{n-1} (q_{j-1} - q_j)xp_j.$$

Let  $a_0$  be some fixed lift of  $b$ . Then

$$E = \{Ta_0 + Ti \mid i \in I\}, \quad \overline{E} = \{Ta_0 + x \mid x \in \overline{TI}\}.$$

Since  $T$  is an elementary operator,  $\overline{TI}$  is proximal in  $\overline{TM(I)}$  by Theorem 4 .

This implies that in  $\overline{E}$  there is an element of norm 1. Indeed, since  $\overline{E}$  is the set of all lifts of the element  $Ta_0 + \overline{TI} \in \overline{TM(I)}/\overline{TI}$ , it is enough to prove that the norm  $\|Ta_0 + \overline{TI}\|$  of the element  $Ta_0 + \overline{TI}$  in  $\overline{TM(I)}/\overline{TI}$  is equal to 1. We have

$$\|Ta_0 + \overline{TI}\| = \inf_{x \in \overline{TI}} \|Ta_0 + x\| \leq \inf_{i \in I} \|Ta_0 + Ti\| = \inf_{x \in E} \|x\| = 1$$

by Theorem 12.1.6 of [1]. It is obvious that  $\|Ta_0 + \overline{TI}\| \geq 1$  because  $(Ta_0 + x)$  is a lift of  $b$  for any  $x \in \overline{TI}$  whence

$$\|Ta_0 + \overline{TI}\| = \inf_{x \in \overline{TI}} \|Ta_0 + x\| \geq \|b\| = 1.$$

Thus in  $\overline{E}$  there is an element of norm 1, that is a nilpotent lift of  $b$  of norm 1.  $\square$

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